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When is a power series ring n-root closed?

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Abstract

Given commutative rings $A \subseteq B$, we present a necessary and sufficient condition for the power series ring $A[[\mathbf{X}]]$ to be n-root closed in $B[[\mathbf{X}]]$. This result leads to a criterion for the the power series ring $A[[\mathbf{X}]]$ over an integral domain A to be n-root closed (in its quotient field). For a domain A, we prove: if A is Mori (for example, Noetherian), then $A[[\mathbf{X}]]$ is n-root closed iff A is n-root closed; if A is Prüfer, then $A[[\mathbf{X}]]$ is root closed iff A is completely integrally closed.

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0. Introduction

All rings considered below are commutative with identity and all ring homomorphisms preserve the identity. Henceforth, let $A \subseteq B$ be rings (with the same identity). As usual, if $n \geq 1$ is an integer we say that A is *n-root closed* in B if $b^n \in A$ with $b \in B$ implies $b \in A$. Also, A is said to be *root closed* in B if A is n-root closed in B for all $n \geq 1$. Throughout this paper, we will formulate definitions for pairs of rings $A \subseteq B$ as above. If A is an integral domain, and no ring B is mentioned, we intend B to be the quotient field of A. For example, a domain is said to be *root closed* if it is root closed in its quotient field.

The main purpose of this paper is to characterize the property of n-root closure in power series rings in the above context of pairs of rings, in order to provide a unified

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approach to results in the literature on power series rings over integral domains and von Neumann regular rings and in order to obtain new results on this subject.

Much of the literature on n-root closure has focused on connections with seminormality (cf. [1–3, 5, 6, 10]); the emphasis on von Neumann regularity in [25] is a notable exception (see also Theorem 1.27 below). As in [9], the ring A is said to be *seminormal* in B if $b \in B$, $b^2 \in A$, $b^3 \in A$ implies $b \in A$; equivalently, if $b \in B, b^i \in A$ for all sufficiently large i implies $b \in A$. If A is n-root closed in B for some $n \geq 2$, then A is seminormal in B, but the converse is false.

We denote by $\mathbf{X}$ a nonempty set of indeterminates, and define the ring $A[[\mathbf{X}]]$ as the union of all rings $A[[\mathbf{Y}]]$, where $\mathbf{Y}$ is a finite subset of $\mathbf{X}$ [15, (1.1)].

It is known that $A[X]$ is n-root closed (resp., seminormal) in $B[X]$ if and only if A is n-root closed (resp., seminormal) in B [10]. Moreover, for an integral domain A with quotient field K, the domain $A[X]$ is n-root closed (resp., seminormal) if and only if A is n-root closed (resp., seminormal) since $K[X]$ is integrally closed in its quotient field $K(X)$. However, the analogous "n-root closed" assertion for rings of formal power series does not hold. Indeed, an example of Ohm [20, Example 2.1] shows that A integrally closed and Archimedean does not imply that $A[[X]]$ is root closed. Recall that a domain A is *Archimedean* if $\bigcap_{i=1}^{\infty} a^i A = (0)$ for any noninvertible element $a \in A$ [24]. Examples of Archimedean domains include Noetherian domains, one-dimensional domains, and completely integrally closed domains.

Observe that $\mathbb{Z}[[\mathbf{X}]]$ is completely integrally closed, and hence root closed. However, $\mathbb{Z}[[\mathbf{X}]]$ is not n-root closed in $\mathbb{Q}[[\mathbf{X}]]$ for any $n \geq 2$, even though $\mathbb{Z}$ is n-root closed in $\mathbb{Q}$ for all n [25, Example 1] (cf. Proposition 1.28). On the positive side, $A[[\mathbf{X}]]$ is seminormal in $B[[\mathbf{X}]]$ if and only if A is seminormal in B; if A is a domain, then $A[[\mathbf{X}]]$ is seminormal iff A is seminormal [11].

Recall from [1] that $\mathscr{C}(A,B)$ is the set of all integers $n \geq 2$ for which A is n-root closed in B (actually, $n \geq 1$ in [1]). Thus $\mathscr{C}(A,B)$ is a subsemigroup of the multiplicative semigroup of positive integers and it is generated by primes, since A is mn-root closed in B iff A is both m- and n-root closed in B. If A is a domain with quotient field K, by our convention, we define $\mathscr{C}(A)$ to be $\mathscr{C}(A,K)$. Recall from [1] that if P is any set of primes, then there is a domain A such that $\mathscr{C}(A)$ is generated by P.

We characterize in Section 1 the property

$$A[[\mathbf{X}]] \text{ is } n\text{-root closed in } B[[\mathbf{X}]]$$

and also the property

$$\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A,B).$$

Since $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) \subseteq \mathscr{C}(A,B)$ in general, the latter property means that $A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$ for each n such that A is n-root closed in B.

To show that if the rings $A \subseteq B$ satisfy our characterization, then $A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$, we use Seidenberg's proof of the fact that for a domain A, if $A[[X]]$ is n-root closed and n is invertible in A, then A is Archimedean [23, Theorem, p. 171] (see also [20, footnote 2, p. 321] and Proposition 2.11). For the converse direction in the characterization, we need some preliminary results on seminormality (Lemma 1.1). These results are also used in Section 3 to obtain a characterization of the complete integral closure of $A[[\mathbf{X}]]$ for a seminormal domain A (Theorem 3.3).

Recall that the complete integral closure of a domain A (in its quotient field K) is

$$A^* = \{t \in K \mid at^i \in K \text{ for some nonzero } a \in A \text{ and all } i \geq 1\}.$$

In Section 2, we deal with root closure properties of power series *domains*. If A is a domain, then $A[[\mathbf{X}]]$ is n-root closed (in its quotient field) iff $A[[\mathbf{X}]]$ is n-root closed in $A^*[[\mathbf{X}]]$ (Proposition 2.3(2)); this fact enables us to apply to domains the results of Section 1. We prove that if A is Mori (for example, Noetherian) and n-root closed, then $A[[\mathbf{X}]]$ is also n-root closed (Theorem 2.13). Recall that a domain is *Mori* if it satisfies the ascending chain condition on integral divisorial ideals [21]. As we show in Corollary 2.8, the Mori assumption can be replaced in the above result by the assumption that A is a power series ring over a domain. Also, if A is a Prüfer domain and n is invertible in A, then $A[[\mathbf{X}]]$ is n-root closed if and only if A is completely integrally closed (Theorem 2.16).

In the above-mentioned result on Prüfer domains and in other results, we do not have to assume that n is invertible in A, but rather that the set $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.

Thus, given rings $A \subseteq B$, we will use the following property:

$$\text{The set } \mathscr{C}(A,B) \text{ generates the ideal } A. \tag{1}$$

This means that either A is m-root closed in B for some $m \geq 2$ invertible in A, or that A is p-root closed in B for two distinct primes p. Note that if A is a domain of finite characteristic and B is a ring containing A, then this property is satisfied iff A is p-root closed in B for a prime $p \neq \operatorname{char} A$.

Property (1) is useful since property $\mathscr{P}_n(A,B)$, defined in Section 1 and used for characterizing n-root closure of power series rings, is well behaved in terms of ideals. That is, given $A \subseteq B$, the set $S = \{n \mid \mathscr{P}_n(A,B) \text{ is satisfied}\}$ is the set of all positive integers in the ideal generated by S in A (Proposition 1.4). This is rather surprising since n-root closure behaves differently.

We show that if A is root closed in B and the pair $(A[[\mathbf{X}]], B[[\mathbf{X}]])$ satisfies property (1), then $A[[\mathbf{X}]]$ is root closed in $B[[\mathbf{X}]]$; more generally, if the pair $(A[[\mathbf{X}]], B[[\mathbf{X}]])$ satisfies property (1), then $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A,B)$ (Corollary 1.25).

The elements a and b of A are called *associated* (in A) if $a = ub$ for some unit $u \in A$. The set of all units of A is denoted by $\mathscr{U}(A)$.

For any nonempty subset S of B and element $c \in B$, we use the notation

$$\langle S : c\rangle_B = \{b \in B \mid cb^i \in S \text{ for all } i \geq 1\}.$$

If A is an integral domain with quotient field K and $S \subseteq K$, then, by definition $\langle S : c\rangle = \langle S : c\rangle_K$. If in addition $c \neq 0$ and $S \subseteq A$, then clearly $\langle S : c\rangle \subseteq A^*$.

As usual, we let $(S : c)_B = \{b \in B \mid cb \in S\}$.

1. Root closure in power series over pairs of rings

We begin by recording some useful properties of $\langle A : c\rangle_B$.

Lemma 1.1. *Let $A \subseteq B$ be rings such that A is seminormal in B, and let $c, c_1, c_2 \in A$. Then*

(i) $\langle A : c^m\rangle_B = \langle A : c\rangle_B$ *for any* $m \geq 1$.
(ii) *If* $\sqrt{Ac_1} = \sqrt{Ac_2}$, *then* $\langle A : c_1\rangle_B = \langle A : c_2\rangle_B$.
(iii) $\langle\langle A : c_1\rangle_B : c_2\rangle_B = \langle A : c_1c_2\rangle_B$.
(iv) $\langle\langle A : c\rangle_B : c\rangle_B = \langle A : c\rangle_B$.
(v) $\langle A : c\rangle_B$ *is an A-subalgebra of B.*

Proof.

(i) Clearly $\langle A : c\rangle_B \subseteq \langle A : c^m\rangle_B$. Conversely, let $b \in \langle A : c^m\rangle_B$. Let $k \geq 1$ be minimal with respect to the property that $c^k b^i \in A$ for all $i \geq 1$. If $k > 1$, then $(c^{k-1}b^i)^2 = c^{k-2}(c^k b^{2i}) \in A$ and $(c^{k-1}b^i)^3 = c^{2k-3}(c^k b^{3i}) \in A$ for all i. Since A is seminormal in B, we obtain $c^{k-1}b^i \in A$ for all i, a contradiction.
(ii) By hypothesis, there is a positive integer m such that $c_1^m \in Ac_2$. Hence $\langle A : c_2\rangle_B \subseteq \langle A : c_1^m\rangle_B = \langle A : c_1\rangle_B$ by part (i). Similarly, $\langle A : c_1\rangle_B \subseteq \langle A : c_2\rangle_B$.
(iii) Let $b \in B$. We have $b \in \langle\langle A : c_1\rangle_B : c_2\rangle_B$ iff $c_2 b^i \in \langle A : c_1\rangle_B$ for all $i \geq 1$, that is, $c_1(c_2 b^i)^j \in A$ for all $i \geq 1$ and $j \geq 1$. Thus $\langle A : c_1c_2\rangle_B \subseteq \langle\langle A : c_1\rangle_B : c_2\rangle_B$. On the other hand, by taking $j = 1$, we obtain the reverse inclusion.
(iv) This follows by taking $c_1 = c_2 = c$ in part (iii) and then applying part (i).
(v) If $b, b' \in \langle A : c\rangle_B$, then $b + b', bb' \in \langle A : c^2\rangle_B = \langle A : c\rangle_B$ by part (i). □

Remark 1.2. If $A \subseteq B$ are rings such that A is seminormal in B and $c \in A$, then $\langle A : c\rangle_B$ is the largest subring of $(A : c)_B$, and $\langle A : c\rangle_B$ is seminormal in B.

Let $A \subseteq B$ be rings and let $n \geq 1$. We define the following property, which will be used to characterize n-root closure in power series rings:

$\mathscr{P}_n(A,B)$ If $b \in B, a \in A$ and $nab \in A$, then $nab^2 \in A$.

We first deal with property $\mathscr{P}_n(A,B)$ in order to apply this later to root closure.

Remark 1.3. (1) Property $\mathscr{P}_n(A,B)$ means that if $b \in B$, if $c \in An$ and $cb \in A$, then $cb^2 \in A$. In particular, if m and n are positive integers such that $m \mid n$ in A, then property $\mathscr{P}_m(A,B)$ implies property $\mathscr{P}_n(A,B)$.

(2) Let $A \subseteq B \subseteq C$ be rings and $n \geq 1$. Then property $\mathscr{P}_n(A,C)$ implies property $\mathscr{P}_n(A,B)$.

(3) It is easy to show that $\mathscr{P}_n(A,B)$ is a local property: property $\mathscr{P}_n(A,B)$ holds iff property $\mathscr{P}_n(A_S,B_S)$ holds for every multiplicative subset S of A (alternatively, property $\mathscr{P}_n(A_M,B_{A\setminus M})$ holds for every maximal ideal M of A).

Note that n-root closure is also a local property, but contrast Remark 1.3(3) with Remark 2.1.

Proposition 1.4. *Let $A \subseteq B$ be rings and*

$$S := \{n \geq 1 \mid \text{property } \mathscr{P}_n(A,B) \text{ is satisfied}\}.$$

Then $S = \{n \geq 1 \mid n \cdot 1 \in (S)A\}$, the set of positive integers in the ideal generated by S in A.

Proof. Let I be the ideal generated by S in A. Let d be a positive integer in I. Thus we can write $d = \sum_{i=1}^{r} n_i a_i$, where n_i are positive integers such that property $\mathscr{P}_{n_i}(A,B)$ is satisfied and $a_i \in A$ for each i. Let d' be a positive integer satisfying $An_1 + \cdots + An_r = Ad'$. Thus $d' = \sum_{i=1}^{r} n_i a'_i$ for some elements a'_i in A. Assume that $b \in B$, $a \in A$ and $d'ab \in A$. For each i, we have $n_i ab \in A$. Thus each $n_i ab^2 \in A$ by property $\mathscr{P}_{n_i}(A,B)$, and hence $d'ab^2 = \sum_{i=1}^{r} a'_i(n_i ab^2) \in A$. Thus property $\mathscr{P}_{d'}(A,B)$ is satisfied, and hence so is property $P_d(A,B)$ by Remark 1.3(1). □

In connection with Proposition 1.4, note that if A is a ring and S is a set of positive integers, then a positive integer d generates the ideal (S) of A if and only if d is associated in A to the greatest common divisor g in $\mathbb{Z}$ of the integers in S. If d and g are not zero divisors in A, this means that both d/g and g/d are in A.

We shall repeatedly use Proposition 1.4, especially in the form detailed in the next remark.

Remark 1.5. Let $A \subseteq B$ be rings. Then:

(i) If m and n are positive integers which are associated in A, then properties $\mathscr{P}_m(A,B)$ and $\mathscr{P}_n(A,B)$ are equivalent.

(ii) If n is invertible in A, then property $\mathscr{P}_n(A,B)$ is equivalent to property $\mathscr{P}_1(A,B)$.

(iii) Property $\mathscr{P}_1(A,B)$ implies property $\mathscr{P}_n(A,B)$ for all $n \geq 1$.

(iv) If m and n are relatively prime positive integers, then properties $\mathscr{P}_m(A,B)$ and $\mathscr{P}_n(A,B)$ hold if and only if property $\mathscr{P}_1(A,B)$ holds.

(v) If $n = 0$ in A, then property $\mathscr{P}_n(A,B)$ holds.

Lemma 1.6. *Let $A \subseteq B$ be rings and $n \geq 1$ such that property $\mathscr{P}_n(A,B)$ holds. Let $a \in A, b \in B$ such that $nab \in A$. Then $n^i ab^i, 2^{i-1} nab^i \in A$ for all $i \geq 1$.*

Proof. We use induction on i starting with $i = 1$. Let $i > 1$.

By the inductive assumption, $n^{i-2}ab^{i-2}$ and $n(n^{i-2}ab^{i-2})b = n^{i-1}ab^{i-1}$ are in A. By property $\mathscr{P}_n(A,B)$, we obtain that $n^i ab^i = n[n(n^{i-2}ab^{i-2})b^2] \in A$.

On the other hand, we obtain by the inductive assumption that $na(b+2^{i-2}b^{i-1}) \in A$, and by property $\mathscr{P}_n(A,B)$ therefore $na(b+2^{i-2}b^{i-1})^2 = na(b^2+2^{i-1}b^i+(2^{i-2}b^{i-1})^2) \in A$. Since nab^2 and $na(2^{i-2}b^{i-1})^2$ arc in A, we conclude that $2^{i-1}nab^i \in A$. □

Proposition 1.7. *Let $A \subseteq B$ be rings and $n \geq 1$. If either n is odd or A is seminormal in B, then the following two conditions are equivalent:*
(1) *Property $\mathscr{P}_n(A,B)$ holds.*
(2) *If $a \in A, b \in B$ and $nab \in A$, then $nab^i \in A$ for all $i \geq 1$.*

Proof. We need only show (1) $\Rightarrow$ (2). If n is odd, this follows from Lemma 1.6.
Now assume that A is seminormal in B. Let $a \in A$, $b \in B$, and $nab \in A$. Set $c = na$. Using property $\mathscr{P}_n(A,B)$, we obtain by induction on i that $cb^{2^i} \in A$ for all $i \geq 1$. Let $m \geq 1$. Choose $i > 1$ such that $2^{i-1} > m$. Then $(cb^m)^{2^i} = c^{2^i - m}(cb^{2^i})^m \in A$ and $(cb^m)^{2^i+1} = c^{2^i+1-2m}(cb)^m(cb^{2^i})^m \in A$. Since A is seminormal in B, we conclude that $cb^m \in A$. $\square$

Proposition 1.7 does not hold in general for n even (see Example 2.20 below).

Remark 1.8. Condition (2) in Proposition 1.7 can be reformulated as follows:

$$\text{If } a \in A, \text{then } \langle A : na \rangle_B = (A : na)_B. \qquad (2')$$

Moreover, if A is seminormal in B, then this condition is also equivalent to the following:

$$\text{If } a \in A,\ b \in B \text{ and } ab \in A, \text{then } nab^i \in A \text{ for all } i \geq 1. \qquad (2'')$$

For a proof, by Proposition 1.7, we need only show that if A is seminormal in B, then (2″) implies property $\mathscr{P}_n(A,B)$. Assume condition (2″). Let $a \in A$, $b \in B$, and $nab \in A$. Since $(na)b \in A$, we obtain by (2″) that $n(na)b^i \in A$ for all $i \geq 1$. Thus $(nab^2)^2$ and $(nab^2)^3$ are in A. Since A is seminormal in B, we conclude that $nab^2 \in A$, and hence property $\mathscr{P}_n(A,B)$ holds. (Alternatively, since $b \in \langle A : n^2a \rangle_B$, we obtain by Lemma 1.1 (ii) that $b \in \langle A : na \rangle_B$, that is, condition (2′) holds.)
From Remark 1.8 and Proposition 1.4 we easily obtain:

Remark 1.9. If $A \subseteq B$ are rings such that A is seminormal in B, then the ideal generated in $\mathbb{Z}$ by the set $S := \{n \geq 1 \mid \text{property } \mathscr{P}_n(A,B) \text{ is satisfied}\}$ is a radical ideal.

Proposition 1.10. *Let $A \subseteq B$ be rings satisfying property $\mathscr{P}_1(A,B)$. Then*

$$\mathscr{U}(A) = \mathscr{U}(B) \cap A.$$

Proof. Let $a \in \mathscr{U}(B) \cap A$. Since $a(1/a) \in A$, we obtain that $1/a = a(1/a)^2 \in A$. $\square$

Corollary 1.11. *Let $A \subseteq B$ be rings. If B is a field, then property $\mathscr{P}_1(A,B)$ holds iff A is a field.*

Proposition 1.12. *Let $A \subseteq B$ be rings. Assume that A is a domain and that there exists a ring containing B and the quotient field K of A. For any $n \geq 1$, property $\mathscr{P}_n(A,B)$ holds iff property $\mathscr{P}_n(A, K \cap B)$ holds.*

Proof. If property $\mathscr{P}_n(A,B)$ holds, then property $\mathscr{P}_n(A,K\cap B)$ also holds by Remark 1.3(2). Conversely, assume that property $\mathscr{P}_n(A,K\cap B)$ holds. Let $a\in A$ such that $na\neq 0$, and let $b\in B$ such that $nab\in A$. Then $b\in K\cap B$. Hence $nab^2\in A$ and property $\mathscr{P}_n(A,B)$ holds. □

The proof of the next proposition is straightforward.

Proposition 1.13. *Let $A\subseteq B$ be rings with a common ideal I. For any positive integer n, the properties $\mathscr{P}_n(A,B)$ and $\mathscr{P}_n(A/I,B/I)$ are equivalent.*

We recall that a ring A is *p-injective* iff each principal ideal of A is an annihilator of some subset of A (see [17, 18]). Equivalently, this means that $(0:(0:a)_A)_A=Aa$ for each $a\in A$; another equivalent property is the following: if $a,c\in A$ and $(0:a)_A\subseteq(0:c)_A$, then $c\in Aa$. (Indeed, if A is p-injective and $a,c\in A$ with $(0:a)_A\subseteq(0:c)_A$, then $c\in(0:(0:c)_A)_A\subseteq(0:(0:a)_A)_A=Aa$. For the converse, let $a\in A$ and let $c\in(0:(0:a)_A)_A$. Thus $(0:a)_A\subseteq(0:c)_A$ and by assumption, $c\in Aa$. Thus $(0:(0:a)_A)_A=Aa$, and A is p-injective.) Note that a reduced ring is p-injective iff it is zero dimensional, that is, von Neumann regular. (The fact that a von Neumann regular ring is p-injective is immediate. For the converse, let A be a reduced p-injective ring and let $a\in A$. Since A is reduced, we have $(0:a)_A=(0:a^2)_A$, and since A is p-injective we obtain $Aa=Aa^2$. Thus A is von Neumann regular. Cf. [18, Proposition 1].) However, a zero-dimensional ring is not necessarily p-injective (for example, $k[X,Y]/(X^2,XY,Y^2)$ for k a field). Moreover, the ring $\prod_{i=1}^{\infty}\mathbb{Z}/2^i\mathbb{Z}$ is p-injective, being a product of p-injective rings, but it has infinite Krull dimension [16].

Proposition 1.14. *Let A be any ring. Then property $\mathscr{P}_1(A,B)$ is satisfied for any ring extension B of A $\Leftrightarrow$ A is p-injective.*

Proof. ($\Leftarrow$): Suppose that A is p-injective. Let B be a ring extension of A, $a\in A$, $b\in B$ and $ab=c\in A$. Then $(0:a)_A\subseteq(0:c)_A$. By p-injectivity, $c\in Aa$, so, $ab^2=cb\in Aab\subseteq A$. Hence property $\mathscr{P}_1(A,B)$ is satisfied.

($\Rightarrow$): Let $a,c\in A$ such that $(0:a)_A\subseteq(0:c)_A$. We have to show that $c\in Aa$. Let T be an indeterminate over A, and set $B=A[T]/(aT-c)$. We first show that $A\cap(aT-c)A[T]=(0)$. Indeed, let $d\in A\cap(aT-c)A[T]$. For some $g(T)=\sum_{i=0}^{m}g_iT^i\in A[T]$, we have $d=(aT-c)g$. If $g\neq 0$, we may assume that $m\geq 0$ is minimal. We have $g_ma=0$; hence $g_mc=0$. If $m\geq 1$, then $d=(aT-c)\sum_{i=0}^{m-1}g_iT^i$, a contradiction. Thus $m=0$, and $d=(aT-c)g_0=0$. Hence we have a canonical embedding $A\subseteq B=A[t]$, where t is the canonical image of T. By property $\mathscr{P}_1(A,B)$, we have $ct=at^2\in A$ since $at=c\in A$. By the previous argument, $cT=e+(aT-c)g_0$ for some $e,g_0\in A$. Thus $c=ag_0\in Aa$, as claimed. □

Corollary 1.15. *Let A be a reduced ring. Then property $\mathscr{P}_1(A,B)$ is satisfied for every ring extension B of A $\Leftrightarrow$ A is von Neumann regular.*

In particular, if A is a field, then property $\mathscr{P}_1(A,B)$ holds for every ring extension B of A.

The fact that a von Neumann regular ring A satisfies property $\mathscr{P}_1(A,B)$ for each ring extension B of A is easily proved directly. Indeed, if A is von Neumann regular and if B is a ring extension of A, $a \in A$, $b \in B$ and $ab \in A$, then $ab^2 \in Aa^2b^2 = A(ab)^2 \subseteq A$. Thus property $\mathscr{P}_1(A,B)$ is satisfied.

Proposition 1.16. *Assume that $A \subseteq B$ are rings such that A is seminormal in B and let $n \geq 1$. Then properties $\mathscr{P}_n(A,B)$ and $\mathscr{P}_n(A[[\mathbf{X}]],B[[\mathbf{X}]])$ are equivalent.*

Proof. If property $\mathscr{P}_n(A[[\mathbf{X}]],B[[\mathbf{X}]])$ is satisfied, then property $\mathscr{P}_n(A,B)$ is also satisfied, even if A is not seminormal in B.

Conversely, assume that property $\mathscr{P}_n(A,B)$ is satisfied. By [11], if a ring D is seminormal in a ring extension E, then $D[[\mathbf{X}]]$ is seminormal in $E[[\mathbf{X}]]$. Also, property $\mathscr{P}_n(A[[\mathbf{X}]],B[[\mathbf{X}]])$ is satisfied iff property $\mathscr{P}_n(A[[\mathbf{Y}]],B[[\mathbf{Y}]])$ is satisfied for every finite subset $\mathbf{Y}$ of $\mathbf{X}$. Thus it is enough to prove the proposition in case $\mathbf{X}$ contains just one indeterminate X. Let $h \in A[[X]], f \in B[[X]]$ such that $nhf \in A[[X]]$. Write $f(X) = \sum_{i=0}^{\infty} b_iX^i$. By factoring out a power of X, we may assume that $c := h(0) \neq 0$. We then have $nb_ic^{i+1} \in A$ for all $i \geq 0$. By property $\mathscr{P}_n(A,B)$ and Proposition 1.7, $b_i \in \langle A : nc^{i+1}\rangle_B$ for all i, and by Lemma 1.1, $ncA[\{b_i \mid i \geq 0\}] \subseteq A$. We thus have $n(h-h(0))f \in A[[X]]$. By induction, we obtain that $naA[\{b_i \mid i \geq 0\}] \subseteq A$ for all coefficients a of h. Hence $nhf^i \in A[[X]]$ for all i. We conclude that property $\mathscr{P}_n(A[[X]],B[[X]])$ is satisfied. □

Lemma 1.17. *Let n,m,k be integers ≥ 2, and let $A \subseteq B$ be rings such that A is seminormal in B. Then the following two properties are equivalent:*

(1) *Property $\mathscr{P}_n(A,B)$ is satisfied.*

(2) *If $b \in B, a \in A$ and $n^m a^k b \in A$, then $nab \in A$.*

Proof. (1) ⇒ (2): Let $a \in A$ and $b \in B$ such that $n^m a^k b \in A$. By property $\mathscr{P}_n(A,B)$ and Proposition 1.7, $n^m a^k b^i \in A$ for all $i \geq 1$. For all $j \geq \max(m,k)$, we have $(nab)^j \in A$. Since A is seminormal in B, we conclude that $nab \in A$, and hence that (2) holds.

(2) ⇒ (1): Let $a \in A$ and $b \in B$ such that $nab \in A$. Since $m,k \geq 2$, also $n^m a^k b^2 \in A$. By (2), we obtain $nab^2 \in A$, so property $\mathscr{P}_n(A,B)$ holds. □

By Lemma 1.17, if A is seminormal in B, then property $\mathscr{P}_1(A,B)$ holds iff for $a \in A$, $b \in B$, $a^2b \in A$ implies $ab \in A$ (cf. the definition of property $\mathscr{P}_1(A,B)$).

Theorem 1.18. *Let $A \subseteq B$ be rings and let $n \geq 2$. The following three conditions are equivalent:*

(1) *$A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$.*

(2) *A is n-root closed in B and*

- *If n has at least two distinct prime factors, then property $\mathscr{P}_1(A,B)$ holds.*
- *If n is a power of a prime p, then property $\mathscr{P}_p(A,B)$ holds.*

(3) *A is n-root closed in B and property $\mathscr{P}_p(A,B)$ holds for each prime factor p of n.*

Proof. Clearly, under each one of the three conditions, A is n-root closed, and hence seminormal, in B.

We first prove the equivalence (1) $\Leftrightarrow$ (3). Since a ring D is n-root closed in a ring extension E iff D is p-root closed in E for each prime factor p of n, we may assume that n is prime.

(1) $\Rightarrow$ (3): The ring $A[[X]]$ is n-root closed in $B[[X]]$ for each $X \in \mathbf{X}$, so we may assume that the set $\mathbf{X}$ contains just one indeterminate X.

Now we prove that property $\mathscr{P}_n(A,B)$ holds. Let $a \in A$ and $b \in B$ such that $n^2a^nb \in A$. Let $f(X) = \sqrt[n]{a^n(1+n^2bX)}$, that is, by definition,

$$f(X) = a\sum_{i=0}^{\infty}\binom{1/n}{i}n^{2i}b^iX^i. \tag{3}$$

By Hermite's Theorem [12, Chapter IX, p. 266],

$$\binom{1/n}{i}n^{2i} \in \mathbb{Z} \quad \text{for all } i \geq 0.$$

(Thus, the definition of $f \in B[[X]]$ is meaningful even if $n = 0$ in A. Of course, the case $n = 0$ is immediate from Remark 1.5(v).) We have

$$f^n = a^n(1+n^2bX) \in A[[X]]$$

(see Remark 1.19 below). Thus $f \in A[[X]]$. Since nab is the coefficient of X in $f(X)$, we conclude that $nab \in A$. By Lemma 1.17, property $\mathscr{P}_n(A,B)$ is satisfied.

(3) $\Rightarrow$ (1): We may again assume that the set $\mathbf{X}$ contains just one indeterminate X, since, by induction and Proposition 1.16, we obtain the implication for finitely many indeterminates and then the general case easily follows from the finite one.

Let f be an element in $B[[X]]$ such that $f^n \in A[[X]]$. Write $f(X) = \sum_{i=0}^{\infty} b_iX^i$. Since A is n-root closed in B, $b_0 \in A$. We prove by induction on i that $b_i \in \langle A : nb_0\rangle_B$ for all $i \geq 0$. The coefficient of X in $f^n = (b_0 + b_1X + \cdots)^n$ is $nb_0^{n-1}b_1$. Since $f^n \in A[[X]]$, we obtain $nb_0^{n-1}b_1 \in A$. By property $\mathscr{P}_n(A,B)$ and condition (2′) in Remark 1.8, $b_1 \in \langle A : nb_0^{n-1}\rangle_B$. By Lemma 1.1 (ii), $b_1 \in \langle A : nb_0\rangle_B$. For $i > 1$, we consider the coefficient of X^i in f^n and use the inductive assumption to obtain that $b_i \in \langle A : nb_0\rangle_B$.

Since n is prime, the binomial coefficients $\binom{n}{i}$ for $1 \leq i < n$ are divisible by n in $\mathbf{Z}$. It follows that for $g = (1/X)(f - b_0)$ we have

$$g^n = \left(\frac{1}{X^n}\right)\sum_{i=0}^{n}\binom{n}{i}f^i(-b_0)^{n-i} \in A[[X]],$$

and so, $b_1 = g(0) \in A$. In this way, we inductively obtain that $b_i \in A$ for all i.

For the equivalence (2) $\Leftrightarrow$ (3), we must check just the case that n has at least two distinct prime factors:

(2) $\Rightarrow$ (3): Indeed, property $\mathscr{P}_1(A,B)$ implies property $\mathscr{P}_n(A,B)$ for all n by Remark 1.5(iii).

(3) $\Rightarrow$ (2): Property $\mathscr{P}_1(A,B)$ is satisfied by Remark 1.5(iv). □

Remark 1.19. Generally, formulas like (3) in the proof of Theorem 1.18 are proved using an "abstract nonsense" argument as below.

Let $n \geq 2$ and let A be any ring in which n is invertible. Set

$$B = \mathbb{Z}[1/n][\{b_i \mid i \geq 1\}],$$

where $b_1, b_2, \ldots,$ are independent indeterminates. Set $f(X) = \sum_{i=0}^{\infty} b_i X^i$, where $b_0 = 1$ and

$$g(X) = \left(\sum_{i=0}^{\infty} \binom{1/n}{i} (f(X) - 1))^i \right).$$

Thus f and g are in $B[[X]]$.

For given elements $\tilde{b}_1, \tilde{b}_2, \ldots$ of A, there exists a unique ring-homomorphism ϕ: $B \longrightarrow A$ sending each b_i to $\tilde{b}_i$. We extend this homomorphism to a homomorphism of the power series rings sending $\sum_{i=0}^{\infty} b_i X^i$ to $\sum_{i=0}^{\infty} \tilde{b}_i X^i$. Since this homomorphism is continuous with respect to the X-adic topologies, we see that it is enough to prove that $g^n = f$.

Fix an integer $m \geq 1$. For given real numbers $r_1, \ldots, r_m$ in the interval $(-1,1)$, denote by F and G the real functions obtained from f and g respectively, using the substitution $b_i \rightarrow r_i$ for $1 \leq i \leq m$ and $b_i \rightarrow 0$ for $i > m$. Thus F and G are analytic in the interval $(-1,1)$ and we have $G^n = F$. The coefficient of X^m in $g^n - f$ is a polynomial h_m in $\mathbb{Z}[1/n][b_1, \ldots, b_m]$. We have $h_m(r_1, \ldots, r_m) = 0$ for any real numbers $r_1, \ldots, r_m$ in $(-1,1)$. Thus $h_m = 0$. It follows that $g^n = f$, so formula (3) holds.

Remark 1.20. A particular case of Theorem 1.18 is easy to prove directly: if A has prime characteristic p, then $A[[\mathbf{X}]]$ is p-root closed in $B[[\mathbf{X}]]$ iff A is p-root closed in B.

In Theorem 1.18(3), even if A is root closed in B, it is not sufficient to assume just that property $\mathscr{P}_n(A,B)$ is satisfied. Indeed, let k be a field of finite characteristic p, and set $A = k[X, XY]$, $B = k[X, Y]$. Since A is factorial and B is contained in the quotient field of A, we see that A is root closed in B. Clearly, p is the only prime q for which property $\mathscr{P}_q(A,B)$ holds. Thus, if n is a multiple of p which is not a power of p, then property $\mathscr{P}_n(A,B)$ holds (since $n = 0$ in A), but for each prime factor $q \neq p$ of n, property $\mathscr{P}_q(A,B)$ does not hold. See also Example 2.21.

The following remark will be used in the next section.

Remark 1.21. As a consequence of Theorem 1.18 and its proof, we obtain a further condition, which is equivalent to each of the three conditions of Theorem 1.18:

(4) A is n-root closed in B and for each prime factor p of n and for any nonzero elements $a \in A$ and $b \in B$ such that $p^2 a^p b \in A$, we have $f \in A[[\mathbf{X}]]$, where

$$f(X) = \sqrt[p]{a^p(1 + p^2 bX)} = a \sum_{i=0}^{\infty} \binom{1/p}{i} p^{2i} b^i X^i.$$

We now turn to the consequences of Theorem 1.18.

Corollary 1.22. *For any rings $A \subseteq B$ the following conditions are equivalent:*

(1) $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A,B)$.

(2) • *If $\mathscr{C}(A,B)$ contains at least two distinct primes, then property $\mathscr{P}_1(A,B)$ holds.*
• *If $\mathscr{C}(A,B)$ contains exactly one prime p, then property $\mathscr{P}_p(A,B)$ holds.*

(3) *Property $\mathscr{P}_p(A,B)$ holds for each prime p in $\mathscr{C}(A,B)$.*

Theorem 1.23. *Let $A \subseteq B$ be rings. Then $A[[\mathbf{X}]]$ is root closed in $B[[\mathbf{X}]]$ iff A is root closed in B and property $\mathscr{P}_1(A,B)$ holds.*

Corollary 1.24. *Let $A \subseteq B$ be rings. Let $\mathbf{X}$ and $\mathbf{Y}$ be two nonempty sets of indeterminates over B. Then $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A[[\mathbf{Y}]], B[[\mathbf{Y}]])$. In particular, $A[[\mathbf{X}]]$ is root closed in $B[[\mathbf{X}]]$ if and only $A[[\mathbf{Y}]]$ is root closed in $B[[\mathbf{Y}]]$.*

From the Introduction, we recall that if $A \subseteq B$ are rings, then $\mathscr{C}(A,B)$ generates the ideal A if and only if either A is m-root closed in B for some $m \geq 2$ invertible in A or A is both p- and q-root closed in B for distinct primes p and q.

Corollary 1.25. *Let $A \subseteq B$ be rings. Then the following three conditions are equivalent:*

(1) *$\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.*

(2) *$\mathscr{C}(A,B)$ generates the ideal A and $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A,B)$.*

(3) *$\mathscr{C}(A,B)$ generates the ideal A and property $\mathscr{P}_1(A,B)$ is satisfied.*

In particular, if A is root closed in B, then $A[[\mathbf{X}]]$ is root closed in $B[[\mathbf{X}]]$ iff $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.

Proof. (1) $\Rightarrow$ (2): Assume (1). By Theorem 1.18 and by Proposition 1.4, property $\mathscr{P}_n(A,B)$ holds for all $n \geq 1$. Hence (2) holds by Theorem 1.18.

(2) $\Rightarrow$ (1): Obvious.

The equivalence (1) $\Leftrightarrow$ (3) follows from Theorem 1.18. □

Remark 1.26. (Cf. Proposition 2.3 and Remark 2.14.) If $A \subseteq B$ are rings such that $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) \neq \mathscr{C}(A,B)$, then $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]])$ can contain at most one prime. All these possibilities (one given prime or no primes) can occur, as shown by Examples 2.22 and 2.23.

Theorem 1.27. *Let A be a p-injective ring. Then*

$$\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A, B)$$

for any ring extension B of A.

Proof. The theorem follows from Theorem 1.18 and Proposition 1.14. □

We do not know if the converse of Theorem 1.27 holds.

As a corollary of Theorem 1.27, we obtain Watkins' theorem that if A is von Neumann regular, then $A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$ for any ring extension B of A such that A is n-root closed in B [25, Theorem 1(b) and (c)].

Theorem 1.18 and Proposition 1.10 (with Corollary 1.11) imply

Proposition 1.28. *Let $A \subseteq B$ be rings such that $\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$. Then $\mathscr{U}(A) = \mathscr{U}(B) \cap A$. In particular, if B is a field, then A is a field.*

More generally, for any ring A and multiplicatively closed subset S of nonzero divisors, $\mathscr{C}(A[[\mathbf{X}]], A_S[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$ iff $A = A_S$.

It follows from Proposition 1.28 that if A is a domain with quotient field K and A is not a field, then $A[[\mathbf{X}]]$ can be p-root closed in $K[[\mathbf{X}]]$ for at most one prime p (this holds if $p = \operatorname{char} K$ and A is p-root closed in K); in contrast, recall that if A is n-root closed for some $n \geq 2$, then $A[\mathbf{X}]$ is n-root closed in $K[\mathbf{X}]$.

In the next theorem, we recover and extend a result of Angermüller [5, Theorem 2(b)]:

Theorem 1.29. *Let $A \subseteq B$ be rings. Assume that A is a domain and that there exists a ring containing B and the quotient field K of A. Let $n \geq 2$. Then the following two conditions are equivalent:*

(1) *$A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$.*

(2) *A is n-root closed in B and $A[[\mathbf{X}]]$ is n-root closed in $(K \cap B)[[\mathbf{X}]]$.*

In particular, if $A = K \cap B$ and A is n-root closed in B, then $A[[\mathbf{X}]]$ is n-root closed in $B[[\mathbf{X}]]$.

Theorem 1.29 can be reformulated as follows:

$$\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A, B) \cap \mathscr{C}(A[[\mathbf{X}]], (K \cap B)[[\mathbf{X}]]).$$

Theorem 1.29 can be used to reduce the study of root closure over a pair of domains $A \subseteq B$ to the case that B is an overring of A.

We end this section with some results on pairs of rings with a common ideal in the spirit of [4]. In particular, Proposition 1.30 can be applied to $D + M$ constructions.

Proposition 1.30. (1) *Let $A \subseteq B$ be rings with a common ideal I. Then*

$$\mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) = \mathscr{C}(A/I[[\mathbf{X}]], B/I[[\mathbf{X}]]).$$

(2) *Let $A \subseteq B$ be rings with a common maximal ideal M. Then*

$$\mathscr{C}(A/M, B/M) = \mathscr{C}(A,B) = \mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]).$$

(3) *Let $A \subseteq B$ be domains with a common nonzero maximal ideal M. Then*

$$\mathscr{C}(A[[\mathbf{X}]]) \cap \mathscr{C}(B[[\mathbf{X}]]) = \mathscr{C}(A,B) \cap \mathscr{C}(B[[\mathbf{X}]]).$$

Proof. (1) Since the rings $A[[\mathbf{X}]]$ and $B[[\mathbf{X}]]$ have a common ideal $I[[\mathbf{X}]]$, the assertion follows from [4, Theorem 1.7].

(2) By [4, Theorem 1.7], $\mathscr{C}(A/M, B/M) = \mathscr{C}(A,B)$. By Corollary 1.11, property $\mathscr{P}_1(A/M, B/M)$ holds. By Proposition 1.13, property $\mathscr{P}_1(A,B)$ also holds. Hence $\mathscr{C}(A,B) = \mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]])$ by Corollary 1.22.

(3) Since $M \neq (0)$, the domains $A[[\mathbf{X}]]$ and $B[[\mathbf{X}]]$ have the same quotient field L. By part (2) we obtain $\mathscr{C}(A[[\mathbf{X}]]) \cap \mathscr{C}(B[[\mathbf{X}]]) = \mathscr{C}(A[[\mathbf{X}]], L) \cap \mathscr{C}(B[[\mathbf{X}]], L) = \mathscr{C}(A[[\mathbf{X}]], B[[\mathbf{X}]]) \cap \mathscr{C}(B[[\mathbf{X}]], L) = \mathscr{C}(A,B) \cap \mathscr{C}(B[[\mathbf{X}]])$. □

2. Root closure in power series domains

Let A be a domain and $n \geq 1$. We define property $(\mathscr{P}_n(A))$ as property $\mathscr{P}_n(A, A^*)$, where A^* is the complete integral closure of A. If no misunderstanding is possible, we write $(\mathscr{P}_n)$ instead of $(\mathscr{P}_n(A))$. Property $(\mathscr{P}_n(A))$ should not be confused with property $\mathscr{P}_n(A) = \mathscr{P}_n(A,K)$ (K = the quotient field of A), which is of no interest: indeed, property $\mathscr{P}_n(A) = \mathscr{P}_n(A,K)$ holds iff $A = K$ or $n = 0$ in A.

Remark 2.1. In contrast with property $\mathscr{P}_n(A,B)$ for a pair of rings $A \subseteq B$, property $(\mathscr{P}_n)$ for domains is not a local property (cf. Remark 1.3(3)).

Indeed, let A be a completely integrally closed Prüfer domain of Krull dimension > 1, which contains the rationals (for example, let A be the ring of entire functions, which is an infinite-dimensional completely integrally closed Bézout domain [15, Section 13, Exercises 16–20]). Then A satisfies property $(\mathscr{P}_1)$ by Proposition 2.15 below. Let M be any maximal ideal in A of height > 1. Then A_M is a valuation domain of Krull dimension > 1. By Theorem 2.4 and Corollary 2.17 below, A_M does not satisfy property $(\mathscr{P}_1)$. Since each $n \geq 2$ is invertible in A, we conclude that for all $n \geq 2, A_M$ does not satisfy property $(\mathscr{P}_n)$, although A does.

Note that the domain A is completely integrally closed, but A_M is not (see Corollary 2.17). □

If A is a seminormal domain, then property $(\mathscr{P}_1(A))$ is equivalent to property $(\mathscr{P}_1)$ in [22, Section 1] (see Proposition 1.7 and [22, Lemma 1.2]).

If $A \subseteq B$ are rings and $n \geq 1$, then the *n-root closure of A in B* is the subring of B generated by the set $\{b \in B \mid b^n \in A\}$.

Remark 2.2. If $A \subseteq R \subseteq B$ are rings and $n \geq 1$ such that R contains the n-root closure of A in B, then A is n-root closed in B iff A is n-root closed in R.

The next proposition enables us to treat root closure of power series rings over a domain as a particular case of root closure for power series over pairs of rings studied in Section 1.

Proposition 2.3. *Let A be a domain and $n \geq 2$.*

(1) *A is n-root closed (in its quotient field) iff A is n-root closed in A^*.*

(2) *$A[[\mathbf{X}]]$ is n-root closed iff $A[[\mathbf{X}]]$ is n-root closed in $A^*[[\mathbf{X}]]$.*

Proof. Part (1) immediately follows from Remark 2.2.

For part (2), the "if" assertion follows since $(A[[\mathbf{X}]])^* \subseteq A^*[[\mathbf{X}]]$ [23, Theorem, p. 170]. To prove the "only if" assertion, note that in the notation of Remark 1.21, with $B = A^*$, f belongs to the quotient field of $A[[\mathbf{X}]]$ (indeed, if d is a nonzero element of A such that $db^i \in A$ for all $i \geq 1$, then $df \in A[[\mathbf{X}]]$). Thus $f \in A[[\mathbf{X}]]$ and condition (4) of Remark 1.21 is satisfied, which implies that $A[[\mathbf{X}]]$ is n-root closed in $A^*[[\mathbf{X}]]$. □

As a consequence of Theorem 1.18 and Proposition 2.3, we obtain

Theorem 2.4. *Let A be a domain and let $n \geq 2$. The following three conditions are equivalent:*

(1) *$A[[\mathbf{X}]]$ is n-root closed.*

(2) *A is n-root closed and also:*
- *If n has at least two distinct prime factors, then A satisfies property $(\mathscr{P}_1)$.*
- *If n is a power of a prime p, then A satisfies property $(\mathscr{P}_p)$.*

(3) *A is n-root closed and A satisfies property $(\mathscr{P}_p)$ for each prime factor p of n.*

Now we continue to apply the results of Section 1 to domains. We next reformulate another central theorem, 1.23 (see also Corollary 1.25). First recall that if D is a domain, then $\mathscr{C}(D)$ generates the ideal D if and only if either D is m-root closed for some $m \geq 2$ invertible in D or D is both p- and q-root closed for distinct primes p and q.

Theorem 2.5. *Let A be any domain. Then $A[[\mathbf{X}]]$ is root closed (resp., $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$) iff A is root closed (resp., $\mathscr{C}(A)$ generates the ideal A) and A satisfies property $(\mathscr{P}_1)$.*

Moreover, if A is root closed, then $A[[\mathbf{X}]]$ is root closed if and only if $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.

We do not know if the analogue of the previous theorem holds for integral closure; that is, we ask

Question 2.6. If A is integrally closed and satisfies property $(\mathscr{P}_1)$, is $A[[X]]$ also integrally closed? Equivalently, the question is: if A is integrally closed and $A[[X]]$ is root closed, must $A[[X]]$ be integrally closed?

Note that the analogue of Question 2.6 for rings with zero-divisors has a negative answer: by [25, Example 4] there is a Boolean ring A such that $A[[X]]$ is not integrally closed, although $A[[X]]$ is n-root closed (in its total quotient ring) for all $n \geq 2$. Also note that Proposition 2.3(2) and Theorem 2.4 do not extend to rings with zero-divisors, by [25, Example 6]. (In this example, $B = B^*$ since B is von Neumann regular. For the definition of B^*, see [15, Section 13].)

As a corollary of Theorem 2.4 and Proposition 1.4, we obtain:

Corollary 2.7. *Let A be any domain and let $\mathbf{X}$ and $\mathbf{Y}$ two nonempty sets of indeterminates over A. Then $\mathscr{C}(A[[\mathbf{X}]]) = \mathscr{C}(A[[\mathbf{Y}]])$. In particular, for any domain A, $A[[\mathbf{X}]]$ is root closed if and only $A[[\mathbf{Y}]]$ is root closed.*

Corollary 2.8. *If D is a power series ring over a domain of the type $A = B[[\mathbf{X}]]$, where B is a domain, then $\mathscr{C}(D) = \mathscr{C}(A)$. In particular, D is root closed iff A is root closed.*

Thus if A is a domain such that $A[[X]]$ is root closed, then $A[[X,Y]]$ is also root closed. It is not clear if the analogous property holds for integral closure. This problem is a particular case of Question 2.6.

Proposition 2.9. *Let A be any domain.*

(1) *If A satisfies property $(\mathscr{P}_1)$, then A is Archimedean.*

(2) *If A is completely integrally closed, then A satisfies property $(\mathscr{P}_1)$.*

Proof. (1) Let a be a nonzero element of A such that $\bigcap_{i=0}^{\infty} a^i A \neq (0)$. Thus $1/a \in A^*$ and $a(1/a) = 1 \in A$. By property $(\mathscr{P}_1)$, we have $1/a = a(1/a)^2 \in A$. (Cf. Proposition 1.10.)

(2) Obvious. □

Using the proof of Proposition 2.9(1) with a replaced by na, we obtain the following.

Remark 2.10. If a domain A satisfies property $(\mathscr{P}_n)$, $a \in A$ and na is not invertible in A, then $\bigcap_{i=0}^{\infty} n^i a^i A = (0)$.

In particular, if A satisfies $(\mathscr{P}_n)$ and n is not invertible in A, then $\bigcap_{i=0}^{\infty} n^i A = (0)$.

We are now able to generalize the result of Seidenberg [23, Theorem, p. 171] that was mentioned in the Introduction.

Proposition 2.11. *If A is a domain and $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$, then A is Archimedean.*

Proof. This follows from Theorem 2.4 and Proposition 2.9(1). □

As a consequence of Theorem 2.4, Remark 2.10 and Proposition 1.4, we obtain the following.

Remark 2.12. If $A[[\mathbf{X}]]$ is an n-root closed domain and n is not invertible in A, then $\bigcap_{i=0}^{\infty} n^i A = (0)$.

Theorem 2.13. *Let A be a Mori domain (for example, a Noetherian domain). Then $\mathscr{C}(A[[\mathbf{X}]]) = \mathscr{C}(A)$; that is, for $n \geq 2$, $A[[\mathbf{X}]]$ is n-root closed iff A is n-root closed. In particular, $A[[\mathbf{X}]]$ is root closed iff A is root closed.*

Proof. We may assume that $\mathscr{C}(A) \neq \emptyset$. Thus A is seminormal. By [22, Proposition 1.4], any seminormal Mori domain satisfies $(\mathscr{P}_1)$, and so A satisfies property $(\mathscr{P}_n)$ for all $n \geq 1$ by Remark 1.5(iii). Hence the theorem follows from Theorem 2.4. □

We wonder if A Mori and integrally closed implies $A[[X]]$ integrally closed (see Question 2.6); the answer is affirmative if A is Noetherian (and integrally closed), for $A[[\mathbf{X}]]$ is then completely integrally closed.

Remark 2.14. If A is a domain such that $\mathscr{C}(A[[\mathbf{X}]]) \neq \mathscr{C}(A)$, then $\mathscr{C}(A[[\mathbf{X}]])$ can contain at most one prime. All these possibilities (one given prime or no primes) can occur, as shown by Examples 2.22 and 2.23 below.

Proposition 2.15. *Let A be a Prüfer domain. Then A satisfies property $(\mathscr{P}_1)$ iff A is completely integrally closed.*

Proof. By Proposition 2.9(2), any completely integrally closed domain satisfies property $(\mathscr{P}_1)$. Conversely, assume that the Prüfer domain A satisfies property $(\mathscr{P}_1)$. Let $0 \neq t \in A^*$ and let $a, at \in A$. By [14, Corollary 1.2], there exists a nonzero element $c \in A$ such that $ct, (1-c)/t \in A$. Since

$$\frac{1-c}{t}, \left(\frac{1-c}{t}\right) t \in A,$$

we obtain by property $(\mathscr{P}_1)$ that

$$(1-c)t = \left(\frac{1-c}{t}\right) t^2 \in A.$$

Since also $ct \in A$, we obtain that $t \in A$. Thus A is completely integrally closed. □

It is trivial that if A is a completely integrally closed domain, then $A[[\mathbf{X}]]$ is root closed. We prove the converse in case A is a Prüfer domain.

Theorem 2.16. *Let A be a Prüfer domain. The following conditions are equivalent:*
(1) *$A[[\mathbf{X}]]$ is root closed.*

(2) *$\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.*
(3) *A is completely integrally closed.*

Proof. By Theorem 2.5, $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$ iff $\mathscr{C}(A)$ generates the ideal A and A satisfies property $(\mathscr{P}_1)$. By Proposition 2.15, A satisfies property $(\mathscr{P}_1)$ iff A is completely integrally closed. Also if A is a completely integrally closed domain, then $\mathscr{C}(A)$ generates the ideal A. The theorem follows. □

Note that each of the conditions of Theorem 2.16 implies that A is Archimedean. See Proposition 2.11.

Corollary 2.17. *Let A be a valuation domain. The following conditions are equivalent:*
(1) *$A[[\mathbf{X}]]$ is root closed.*
(2) *$\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.*
(3) *A is completely integrally closed.*
(4) *A is Archimedean.*
(5) $\dim A \leq 1$.

Proof. Indeed, a valuation domain A is completely integrally closed iff $\dim A \leq 1$ [15, Theorem 17.5(3)]. □

It appears to be an open question whether an Archimedean Prüfer domain need be completely integrally closed. (However, see Corollary 2.17 and Proposition 2.18. Of course, unlike the valuation domain case, a completely integrally closed Prüfer domain may have dimension >1.)

Recall that a domain A is a QR-domain if each overring of A is a ring of fractions of A. A QR-domain is necessarily a Prüfer domain, but not conversely. Moreover, Bézout domains can be characterized as the QR-domains that are also GCD-domains.

Proposition 2.18. *Let A be a QR- or a GCD-domain. The following conditions are equivalent:*
(1) *$A[[\mathbf{X}]]$ is root closed.*
(2) *$\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$.*
(3) *A is completely integrally closed.*
(4) *A is Archimedean.*

Proof. Indeed, A is completely integrally closed iff it is Archimedean: see [15, Proposition 27.6] for the case that A is a QR-domain, and [7, Theorem 3.1] if A is a GCD-domain. □

Remark 2.19. (1) By [8, Chapter 1, Theorem 16], a domain A is completely integrally closed iff $A[[\mathbf{X}]]$ is completely integrally closed. Hence in Theorem 2.16, Corollary 2.17 and Proposition 2.18, we may add any of the properties of $A[[\mathbf{X}]]$ intermediate between complete integral closure and the property "$\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$": e.g., integral closure.

(2) It was shown in Theorem 2.16 that if A is a Prüfer domain, then $A[[\mathbf{X}]]$ is root closed if and only if $\mathscr{C}(A[[\mathbf{X}]])$ generates the ideal $A[[\mathbf{X}]]$. However, this equivalence need not hold for an arbitrary domain A. Indeed, Proposition 1.30 may be used to construct a one-dimensional quasilocal domain A such that $\mathscr{C}(A) = \mathscr{C}(A[[\mathbf{X}]]) = \langle P \rangle$ for any given set P of primes.

We end this section with four examples. The first one is related to Proposition 1.7.

Example 2.20. (For a given positive even integer n, a domain A satisfying property $(\mathscr{P}_n)$, but not condition (2) of Proposition 1.7). Let T be an indeterminate over $\mathbb{Z}$. Set

$$A = \mathbb{Z}[nT, nT^2, 2nT^3, \{T^i \mid i \geq 4\}].$$

We have $2n\mathbb{Z}[T] \subseteq A$ and $A^* = \mathbb{Z}[T]$. For any polynomial $g \in \mathbb{Z}[T]$ we have $ng^2 \in A$, since the coefficient of T^3 in g^2 is even. Hence A satisfies property $(\mathscr{P}_n)$. However, we have $nT, nT^2 \in A$, but $nT^3 \notin A$ since n is even. Thus condition (2) of Proposition 1.7 fails for $a = 1, b = T$.

Our next example is related to Proposition 1.4. Note that by this proposition, if $A \subseteq B$ are rings then the set

$$S := \{n \geq 1 \mid \text{property } \mathscr{P}_n(A,B) \text{ is satisfied}\},$$

if it is not empty, equals the set of all positive integers which are multiples of the least positive integer d in S. Moreover, if A is seminormal in B and $S \neq \emptyset$, then by Remark 1.9, d is squarefree. However, the set S can be empty even if A is an integrally closed domain and $B = A^*$ (Example 2.22).

Example 2.21. (For a given positive integer d, a domain A such that the set

$$S := \{n \geq 1 \mid \text{property } (\mathscr{P}_n(A)) \text{ is satisfied}\}$$

equals the set of all positive integers which are divisible by d in Z. Moreover, if d is squarefree, then A can be taken to be root closed). Let Y, Z be independent indeterminates over $\mathbb{Z}$ and set

$$A = \mathbb{Z}[Y, YZ] + d\mathbb{Z}[Y, Z].$$

The assertion on the set S is easily proved.

We now assume that d is squarefree. To show that A is root closed, we may assume that $d > 1$. Let p be a prime factor of d. Let $k = \mathbb{Z}/\mathbb{Z}p$. By [4, Theorem 1.7], it is enough to show that $k[Y, YZ]$ is root closed in $k[Y, Z]$. This follows from the fact that $k[Y, Z]$ is contained in the quotient field of $k[Y, YZ]$ and the ring $k[Y, YZ]$ is isomorphic to $k[Y, Z]$.

Example 2.21 shows, in particular, that, in contrast to the fact that a ring A is mn-root closed if A is both m- and n-root closed, property $(\mathscr{P}_{mn})$ (for a given domain) does not imply properties $(\mathscr{P}_m)$ and $(\mathscr{P}_n)$. See also Example 2.23.

Example 2.22. (An integrally closed domain A such that $A[[\mathbf{X}]]$ is not n-root closed for any $n \geq 2$). Indeed, by Corollary 2.17, any valuation domain A with $\dim A \geq 2$ and containing $\mathbb{Q}$ satisfies our requirements.

For a nonquasilocal example, let $k \subseteq L$ be fields of characteristic zero such that k is algebraically closed in L. Let S and T be two independent indeterminates over L and consider the ring $A = k[S]+TL(S)[T]$. Then A is an integrally closed non-Archimedean domain. On the other hand, $A[[\mathbf{X}]]$ is not n-root closed for any $n \geq 2$. Indeed, assume the contrary for some $n \geq 2$. Then, by Theorem 2.4 and Remark 1.5(ii), A satisfies property $(\mathscr{P}_n)$, and so also $(\mathscr{P}_1)$ since n is invertible in A. Thus A must be Archimedean by Proposition 2.9(1), a contradiction.

Example 2.23. (For a given prime p, a 2-dimensional valuation domain V of zero characteristic such that $V[[X]]$ is p-root closed.

Note that in such an example, V is not completely integrally closed, in particular not Archimedean (cf. Corollary 2.17). Also, by Theorem 2.16, p is the only prime q such that $V[[\mathbf{X}]]$ is q-root closed. By Remark 2.12, we also have $\bigcap_{i=1}^{\infty} p^i V = (0)$). Let V be a valuation domain over $\mathbb{Z}[X]$ in $\mathbb{Q}(X)$ with maximal ideal lying over the ideal (p,X) of $\mathbb{Z}[X]$ and with a prime ideal lying over the ideal $p\mathbb{Z}[X]$ of $\mathbb{Z}[X]$ (cf. [15, Corollary 19.7]). The domain V is uniquely determined by these requirements, namely, $V = \mathbb{Z}[X]_{(X,p)} + P$, where $P = p\mathbb{Z}[X]_{p\mathbb{Z}[X]}$. Indeed, we have the following pullback diagram:

$$\begin{array}{ccc} \mathbb{Z}[X]_{(X,p)} + P & \longrightarrow & \mathbb{Z}[X]_{(p)} \\ \downarrow & & \downarrow \\ (\mathbb{Z}/p\mathbb{Z})[X]_{(X)} & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})(X) \end{array}$$

By [13, Theorem 2.4(1)], $\mathbb{Z}[X]_{(X,p)} + P$ is a valuation domain. Since this valuation domain is dominated by V, we obtain that it equals V.

Clearly V has exactly two nonzero prime ideals: P and XV. Thus V is 2-dimensional. We have $V \subseteq \mathbb{Z}[X]_{p\mathbb{Z}[X]} \subseteq V^*$. Since the domain $\mathbb{Z}[X]_{p\mathbb{Z}[X]}$ is a discrete valuation domain, it is also completely integrally closed, and so it equals V^*. Thus $pV^* \subseteq P \subseteq V$ and this immediately implies that V satisfies property $(\mathscr{P}_p)$.

3. The complete integral closure of $A[[\mathbf{X}]]$

In this section we characterize the complete integral closure of $A[[\mathbf{X}]]$ when A is a seminormal domain (Theorem 3.3 below). We first need two lemmas.

If A is a domain and f is a nonzero element of $A[[X]]$, we denote its first nonzero coefficient by $\mathbf{i}(f)$; we define $\mathbf{i}(0) = 0$.

Lemma 3.1. *Let D be a domain, and let $h \in D[[X]]$. If $\langle D : \mathbf{i}(h)\rangle = D$, then $\langle D[[X]] : h\rangle \subseteq D[[X]]$.*

Proof. By factoring out a power of X, we may assume that $h(0) \neq 0$. Let $f \in \langle D[[X]] : h\rangle$. Thus f is a power series over the quotient field of D; write $f = \sum_{i=0}^{\infty} b_i X^i$. It follows that $hf^i \in D[[X]]$ for all i. Hence $b_0 \in \langle D : h(0)\rangle = D$. Let $g = f - b_0$. We have $hg^i \in D[[X]]$ for all $i \geq 0$. Thus $g \in \langle D[[X]] : h\rangle$. In this way we inductively obtain that $b_i \in D$ for all i. □

Lemma 3.2. *Let A be a seminormal domain with quotient field K. Let $0 \neq h \in A[[X]]$. Set $D = \langle A : \mathbf{i}(h)\rangle$. Then $\langle A[[X]] : h\rangle \subseteq D[[X]]$.*

Proof. By Lemma 1.1(iv), $\langle D : \mathbf{i}(h)\rangle = D$. Thus, $\langle A[[X]] : h\rangle \subseteq \langle D[[X]] : h\rangle \subseteq D[[X]]$ by Lemma 3.1. □

Theorem 3.3. *Let A be a seminormal domain with quotient field K, and let $f \in K[[\mathbf{X}]]$. The following conditions are equivalent:*

(1) $f \in (A[[\mathbf{X}]])^*$.

(2) *There exists a nonzero element $c \in A$ such that $cb^i \in A$ for all coefficients b of f and for all $i \geq 0$.*

(3) *There exists a nonzero element $c \in A$ such that $cB \subseteq A$, where B is the subring of K generated by A and the coefficients of f.*

(4) *There exists a nonzero element $c \in A$ such that $cf^i \in A[[\mathbf{X}]]$ for all $i \geq 0$. Moreover, if A satisfies property $(\mathscr{P}_1)$, then the following condition is also equivalent to the previous ones:*

(5) *$f \in A^*[[\mathbf{X}]]$ and there exists a nonzero element $c \in A$ such that $cf \in A[[\mathbf{X}]]$.*

Proof. First assume that $\mathbf{X}$ contains just one indeterminate.

(1) ⇒ (2): This follows from Lemma 3.2.

(2) ⇒ (3): This follows from Lemma 1.1(v).

(3) ⇒ (4) ⇒ (1): Clear.

Clearly, (2) ⇒ (5).

Next assume that A satisfies property $(\mathscr{P}_1)$ and also condition (5). Hence $cb \in A$ for any coefficient b of f, and by Proposition 1.7 we obtain property (2).

By induction on the number of indeterminates, we obtain the theorem for the case that the set $\mathbf{X}$ is finite. The general case easily follows from the finite case. □

Corollary 3.4. *If A is a seminormal domain satisfying property $(\mathscr{P}_1)$, then*

$$(A[[\mathbf{X}]])^* = A^*[[\mathbf{X}]] \cap A[[\mathbf{X}]]_{A\setminus\{0\}}.$$

Corollary 3.4 can be applied to a seminormal Mori domain since such a domain satisfies property $(\mathscr{P}_1)$.

By Corollary 3.4, if A is a seminormal domain satisfying property $(\mathscr{P}_1)$ and L is the quotient field of $A[[X]]$, then $(A[[\mathbf{X}]])^* = A^*[[\mathbf{X}]] \cap L$. It is not clear if we can replace complete integral closure here by integral closure or by root closure.

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